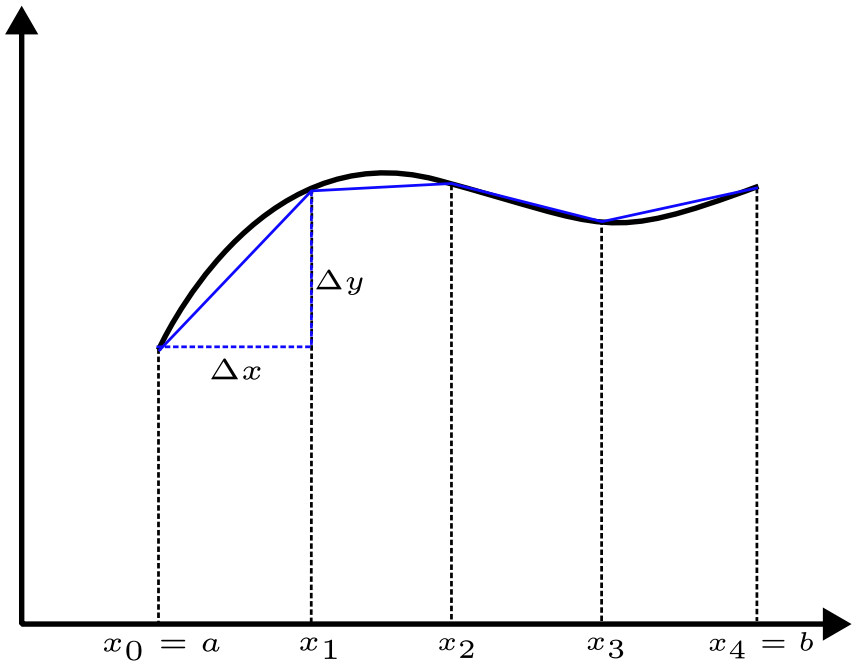
Section 6.5

Estimating Arc Length

Suppose a curve is given by $y = f(x)$ which has a continuous first derivative on $[a, b]$. If one were to walk along the curve starting at the point $(a, f(a))$ and stopping at $(b, f(b))$, how far would one walk? This distance is the *arc length*, L.

Our strategy for estimating the arc length is the same as our strategy for estimating the area under a curve: we divide $[a, b]$ into n subintervals of length $\Delta x = \dfrac{b-a}{n}$, with x_k the right endpoint of the k^{th} subinterval.



We form a piecewise linear curve whose segments connect the points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$. As $n \to \infty$ this piecewise linear curve approximates f and hence its arc length is an approximation of the arc length of f.

$$L \approx \sum_{i=0}^{n-1} \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2} = \sum_{i=0}^{n-1} \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

We then have $L = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$.

It behooves us to write the above summation as a Riemann sum if possible. We then need the summand in the form $g(x)\Delta x$ for some function g.

$$\sum_{i=0}^{n-1} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} =$$

Definition 6.5.1 (Arc length of $y = f(x)$)**.** Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Example 6.5.2. Find the length of the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 4$.

Example 6.5.3. Find the length of the curve $f(x) = x^{2/3}$ between $x = 0$ and $x = 8$.

Theorem 6.5.4 (Arc Length of $g(y)$)**.** Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. The the length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy.$$

If $y = f(x)$ is such that $g(f(x)) = x$ (i.e. $f(x) = g^{-1}(x)$) and f has a continuous first derivative for all x between $g(c)$ and $g(d)$ inclusive, then

$$L = \left| \int_{g(c)}^{g(d)} \sqrt{1 + (f'(x))^2}\, dx \right|.$$

Example 6.5.5. Find the arc length of the curve $x = 2y - 4$ for $-3 \leq y \leq 4$ using calculus and verify with geometry.

Example 6.5.6. Find the length of the curve $x = \dfrac{y^4}{4} + \dfrac{1}{8y^2}$ for $1 \leq y \leq 2$.

Example 6.5.7. Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ on $[-\ln 2, \ln 2]$.

Example 6.5.8. Find the length of the curve $y = f(x) = \ln(x + \sqrt{x^2 - 1})$ on the interval $[0, \sqrt{2}]$.